EARTHQUAKE ENGINEERING AND STRUCTURAL DYNAMICS, VOL. 25, 1139–1149 (1996)

SLIDE ROTATION OF RIGID BODIES SUBJECTED TO A HORIZONTAL GROUND MOTION

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SUMMARY

Simultaneous sliding and rotation of the rigid block is studied, when it is subjected to a strong horizontal motion of the ground. The main purpose of the work is to develop an analytical approach, to predict qualitatively possible behaviour of the body. Two phases of the block rotation are investigated: (1) with a fixed direction of relative sliding, (2) after the friction force changes its direction. The conditions of the overturning for the first phase are formulated. In the second phase the main attention is paid to the analysis of conditions when the body returns back to the initial position and when the final overturning occurs. All the analytical results are compared with numerical calculations.

KEY WORDS: earthquakes; dynamics; rocking; rigid body

INTRODUCTION

Beginning from observations of Japanese tombstones the problem of rocking and overturning of rectangular rigid block, due to a horizontal motion caused by earthquakes, was a subject of many researches. Apparently, Housner[1] was the first who studied strictly rocking without sliding. This case has been also investigated by Yim *et al.*,[2] Spanos and Koh,[3] Hogan,[4,5] Tso and Wong[6] among others. They have shown that the problem under consideration appears to be globally non-linear and have discovered some properties that cannot be inherent in linear systems. The only doubtful assumption for this regime is that the friction coefficient is taken to be so large, to prevent sliding. In practice this coefficient is restricted by a unit value, thus nobody knows if it is sufficient to prevent the sliding or not. Anyway this assumption should be verified after the problem is solved.

Opposite case of perfect sliding (without rotation) has been investigated in the fundamental work of Younis and Tadjbakhsh.[7] This problem is of great interest also in connection with a base isolation of constructions. Particularly it has been proposed that the isolation with a sliding foundation can be very efficient.[8] Some experimental results of Raditchuk and Kislij[9] state wonderful stability of constructions with the sliding basis just in a frequency region of real earthquakes.

It is evident that isolation with absolutely smoothing sliding foundation ($\mu = 0$) defends the construction perfectly, if the ground motion is strictly horizontal. However this situation is non-real, and the friction coefficient cannot usually be reduced lower than $\mu \sim 0{\cdot}1$–$0{\cdot}2$. So it is important to study possible overturning of the construction subjected to a horizontal ground motion in the low friction case.

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CCC 0098–8847/96/101139–11

Received 2 February 1995
Revised 23 April 1996

The possibility of simultaneous rocking and sliding of the body above the horizontally oscillating ground is briefly mentioned in the work of Aslam *et al.*[10] The most advanced numerical investigation has been performed by Ishiyama.[11] He has classified the body motions into six types and has studied the condition of the transformations from one type to another. Also criteria for overturning are formulated numerically. Slide rotation is one of these six types of the possible motions.

At the present work we aim at some analytical results significant, first of all, from the theoretical point of view. They appear to be helpful also in practical aspect. The main attention is paid to a pulse type excitation, contrary to the case when overturning occurs after multiple impacts of the base corners with the ground like in a short-period sinusoidal excitation.

FORMULATION OF THE PROBLEM AND GOVERNING EQUATIONS IN THE FIRST PHASE OF MOTION

Let the rigid rectangular block state on the rigid ground subjected to a horizontal motion caused by an earthquake. The following assumptions are taken into consideration:

(1) the motion of the ground is strictly horizontal;
(2) the values of static and dynamic friction coefficients coincide:

$$\mu_s = \mu_d = \mu, \quad 0 \leqslant \mu \leqslant 1$$

As shown in the work of Younis and Tadjbakhsh for strong earthquakes with a large ground acceleration a_g (that is equivalent to a small value of the friction coefficient μ) the sliding necessarily takes place. For example, if the ground motion is harmonic with respect to time,

$$\dot{x}_g = v_0 \cos \omega t \tag{1}$$

then sliding certainly occurs if

$$\mu < 0{\cdot}537 a_g / g \tag{2}$$

where $a_g = v_0 \omega$. Thus, we accept the third assumption to be realized:

(3) the friction coefficient μ is small enough to provide sliding.

If the body starts to slide the direction of relative sliding between the body and the ground is fixed for a period of time, when we call 'the first phase' of the motion. For this phase the friction force $\bar{F}_f$ does not change its direction, and we have (see Figure 1)

$$F_f = \mu N \tag{3}$$

where $\bar{N}$ is a normal reaction of the ground.

The plane motion of the block is described by the following equations:

$$m\ddot{x}_c = F_f \tag{4a}$$

$$m\ddot{y}_c = N - mg \tag{4b}$$

$$I_c \ddot{\varphi} = F_f y_c - N(a \cos \varphi - h \sin \varphi) \tag{4c}$$

where

$$y_c = a \sin \varphi + h \cos \varphi . \tag{5}$$

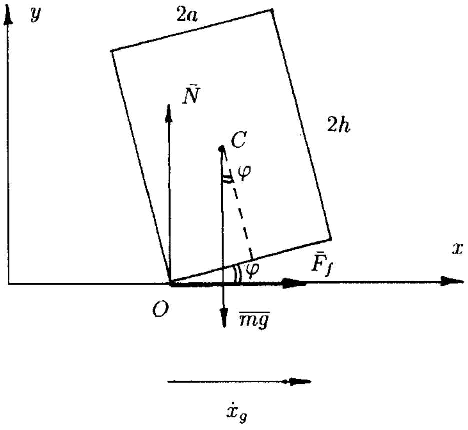

Figure 1. Slide rotation of the body with a fixed direction of the friction force

Here (x, y) is the fixed Cartesian co-ordinate system, $2a$ the width of the block, $2h$ its height. The point above any variable denotes its derivative with respect to the time t. The moment of inertia

$$I_c = \frac{m}{3}(a^2 + h^2) \tag{6}$$

It follows from equations (3)–(5) that

$$N = m(\ddot{y}_c + g) \tag{7a}$$

$$\ddot{y}_c = (a \cos \varphi - h \sin \varphi)\ddot{\varphi} - (a \sin \varphi + h \cos \varphi)\dot{\varphi}^2 \tag{7b}$$

$$I_c\ddot{\varphi} = m(\ddot{y}_c + g)[\mu(a \sin \varphi + h \cos \varphi) - (a \cos \varphi - h \sin \varphi)] \tag{7c}$$

Taking into account equalities (6) and (7b) the last equation can be rewritten as follows:

$$a_1(\varphi)\ddot{\varphi} + b_1(\varphi)\dot{\varphi}^2 = c_1(\varphi) \tag{8}$$

where

$$a_1(\varphi) = 1 + \alpha^2 - 3(\cos \varphi - \alpha \sin \varphi)[(\mu + \alpha) \sin \varphi + (\mu\alpha - 1) \cos \varphi] \tag{9a}$$

$$b_1(\varphi) = 3(\sin \varphi + \alpha \cos \varphi)[(\mu + \alpha) \sin \varphi + (\mu\alpha - 1) \cos \varphi] \tag{9b}$$

$$c_1(\varphi) = 3\frac{g}{a}[(\mu + \alpha) \sin \varphi + (\mu\alpha - 1) \cos \varphi] \tag{9c}$$

The main geometric parameter

$$\alpha = \frac{h}{a} \tag{10}$$

is a relative height of the construction.

The initial conditions

$$t = 0 \,|\, \varphi = 0, \quad \dot{\varphi} = 0 \tag{11}$$

should be added to the equation (8) if the body was initially at rest.

The main goal here is to investigate overturning of the block, that is equivalent to the question if the angle of rotation φ can reach the value $\varphi = \pi/2$. The posed problem is described by equation (8) with initial conditions (11), and so it is non-linear. However for the initial period of motion the non-linear term $\dot{\varphi}^2$ can be rejected, and equation (8) becomes linear. If we set there $a_1(\varphi) \approx a_1(0)$, $c_1(\varphi) \approx c_1(0)$, then the function $\varphi(t)$ can be explicitly determined for small t as

$$\varphi(t) = \frac{3g}{2a} \frac{\mu\alpha - 1}{4 + \alpha(\alpha - 3\mu)} t^2 \tag{12}$$

It follows from equation (12) the well-known result that the condition

$$\mu\alpha - 1 > 0 \sim \mu > \frac{1}{\alpha} \sim \alpha > \frac{1}{\mu} \tag{13}$$

is the necessary and sufficient condition for the body to start its rotation. For further consideration we consider the natural condition (13) to be fulfilled. Hence, the block necessarily will begin to rotate. What is its behaviour later?

Generally, the non-linear equation of the second order (8) can be reduced to a linear equation of the first order. The main ideas of this remarkable transformation were established by Bernoulli. We introduce the new function

$$p(\varphi) = \dot{\varphi} \;\Rightarrow\; \ddot{\varphi} = p(\varphi)p'(\varphi) \tag{14}$$

Then equation (8) can be rewritten as

$$a_1(\varphi)p(\varphi)p'(\varphi) + b_1(\varphi)p^2(\varphi) = c_1(\varphi) \tag{15}$$

After that another function $u_1(\varphi)$ is introduced:

$$u_1(\varphi) = p^2(\varphi) \;\Rightarrow\; u_1'(\varphi) = 2p(\varphi)p'(\varphi) \tag{16}$$

which involves the following linear differential equation of the first order:

$$a_1(\varphi)u_1'(\varphi) + 2b_1(\varphi)u_1(\varphi) = 2c_1(\varphi) \tag{17}$$

with the trivial initial condition

$$\varphi = 0 \,|\, u_1 = 0 \tag{18}$$

Physical meaning of the new function $u_1(\varphi)$ is

$$u_1(\varphi) = \omega^2 \tag{19}$$

where $\omega = \dot{\varphi}(t)$ denotes the angular velocity.

For analysis of the equation (17) qualitative properties of the coefficients $a_1(\varphi)$, $b_1(\varphi)$ and $c_1(\varphi)$ are very important. Obviously these functions are analytical. It can be also proved that they are positive over the interval $\varphi \in (0, \pi/2)$. For the functions $b_1(\varphi)$ and $c_1(\varphi)$ it is rather evident with the natural condition (13). For the function $a_1(\varphi)$ we note that it can be rewritten as

$$a_1(\varphi) = a_1^*(\tau)\cos^2\varphi \tag{20a}$$

where

$$\tau = \tan\varphi,$$

$$a_1^*(\tau) = [1 + \alpha^2 + 3\alpha(\alpha + \mu)]\tau^2 + 3[\alpha(\mu\alpha - 1) - (\mu + \alpha)]\tau + [1 + \alpha^2 - 3(\mu\alpha - 1)] \tag{20b}$$

Determinant of the last binomial,

$$\Delta = (9\mu^2 - 16)(\alpha^2 + 1)^2 < 0 \tag{21}$$

because $\mu < 1$. Therefore, expression (20) is of a constant sign. Besides, as it follows from inequality (13),

$$a_1(0) = 4 + \alpha(\alpha - 3\mu) > 0 \tag{22}$$

so $a_1(\varphi)$ is surely positive.

This result is very important for the analysis of the solution of the equation (17) in the phase space (φ, u_1). It determines that there are no singular points in the phase space, as far as $a_1(\varphi) \neq 0$.

Solution of equation (17) with the trivial initial condition (18) can be obtained by Bernoulli method[12] and is given by the following formula:

$$u_1(\varphi) = 2\xi_1(\varphi)\int_0^{\varphi} \frac{c_1(\psi)}{a_1(\psi)\xi_1(\psi)}\,\mathrm{d}\psi \tag{23}$$

where $\xi_1(\varphi)$ is a solution of the homogeneous equation

$$a_1(\varphi)\xi_1'(\varphi) + 2b_1(\varphi)\xi_1(\varphi) = 0 \tag{24}$$

and has the form

$$\xi_1(\varphi) = \exp\left[-2\int \frac{b_1(\varphi)}{a_1(\varphi)}\,\mathrm{d}\varphi\right] \tag{25}$$

Let us calculate the integral which appeared in the last expression. Introducing the new variable $\tau = \tan\varphi$ it can be rewritten as follows:

$$2\int \frac{b_1(\varphi)}{a_1(\varphi)}\,\mathrm{d}\varphi = 6\int \frac{(\tau + \alpha)[(\mu + \alpha)\tau + (\mu\alpha - 1)]}{(1 + \tau^2)a_1^*(\tau)}\,\mathrm{d}\tau \tag{26}$$

where $a_1^*(\tau)$ is given by formula (20b). Note that

$$6\,\frac{(\tau + \alpha)[(\mu + \alpha)\tau + (\mu\alpha - 1)]}{(1 + \tau^2)a_1^*(\tau)} = \frac{A + B\tau}{a_1^*(\tau)} - \frac{2\tau}{1 + \tau^2} \tag{27a}$$

with

$$A = 6\alpha(\mu\alpha - 1), \qquad B = 2[1 + \alpha^2 + 3\alpha(\alpha + \mu)] \tag{27b}$$

It is well known that the integral of the last two rational functions can be expressed in a closed form. However precise representation does not permit analytical evaluation of integral (23). To come to a closed form for the final result, let us recall that the problem is considered to be under assumption (3) which declares that the friction coefficient μ is rather small. It can be shown that in this case

$$A + B\tau \approx [a_1^*(\tau)]' \tag{28}$$

which yields the following expression for $\xi_1(\varphi)$:

$$\xi_1(\varphi) \approx \frac{1 + \tau^2}{a_1^*(\tau)} = \frac{1}{a_1(\varphi)} \tag{29}$$

A result wonderful of its simplicity. Substituting the last formula to equality (24) we come to the final representation for $u_1(\varphi)$:

$$u_1(\varphi) = \frac{6g}{a} \frac{(\mu\alpha - 1)\sin\varphi + \alpha(1 - \cos\varphi)}{a_1(\varphi)} \tag{30}$$

when μ is small enough. Thorough investigation and comparison with precise numerical calculations has shown that the last formula is a uniform approximation with the relative error not more than 5–10% for all possible values of parameters μ and α. Only for the extremely large $\mu \sim 1$ the error can reach 15–20%. Some results are reflected in Figures 2(a)–(c). Further consideration is based on the formula (31).

Theorem 1. *If the first phase of the motion with the fixed direction of the friction force $\bar{F}_f$ and the condition* (13) *continues for sufficiently long time, the block will be necessarily overturned.*

Proof is based on the formula

$$t_1 = \int_0^{\varphi} \frac{d\varphi}{\sqrt{u_1(\varphi)}} \tag{31}$$

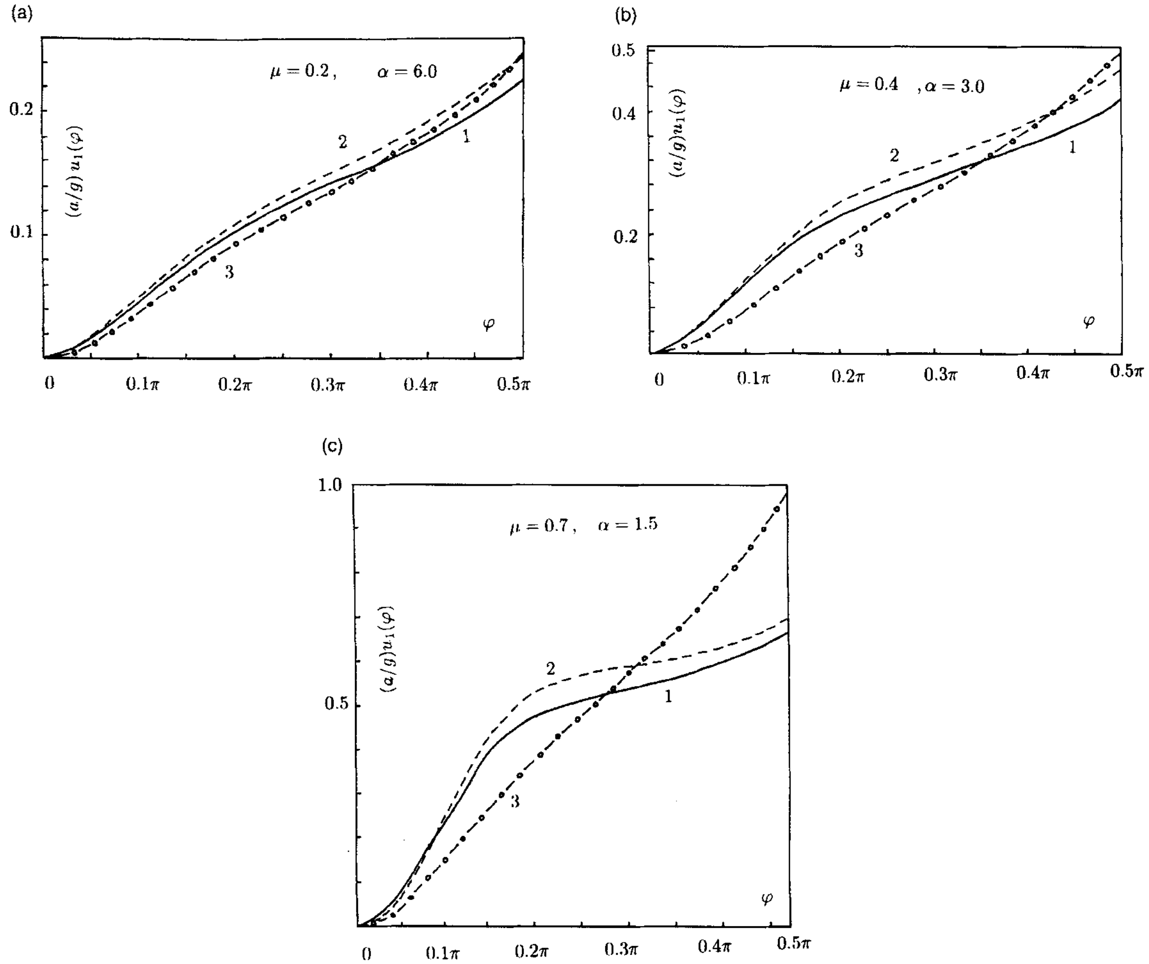

Figure 2. Comparison of exact numerical solution (line 1) for the first phase of motion with the analytical solution (31) (line 2) and solution of a test problem (34) (line 3): (a) $\mu = 0{\cdot}2$; $\alpha = 6{\cdot}0$; (b) $\mu = 0{\cdot}4$; $\alpha = 3{\cdot}0$; (c) $\mu = 0{\cdot}7$; $\alpha = 1{\cdot}5$

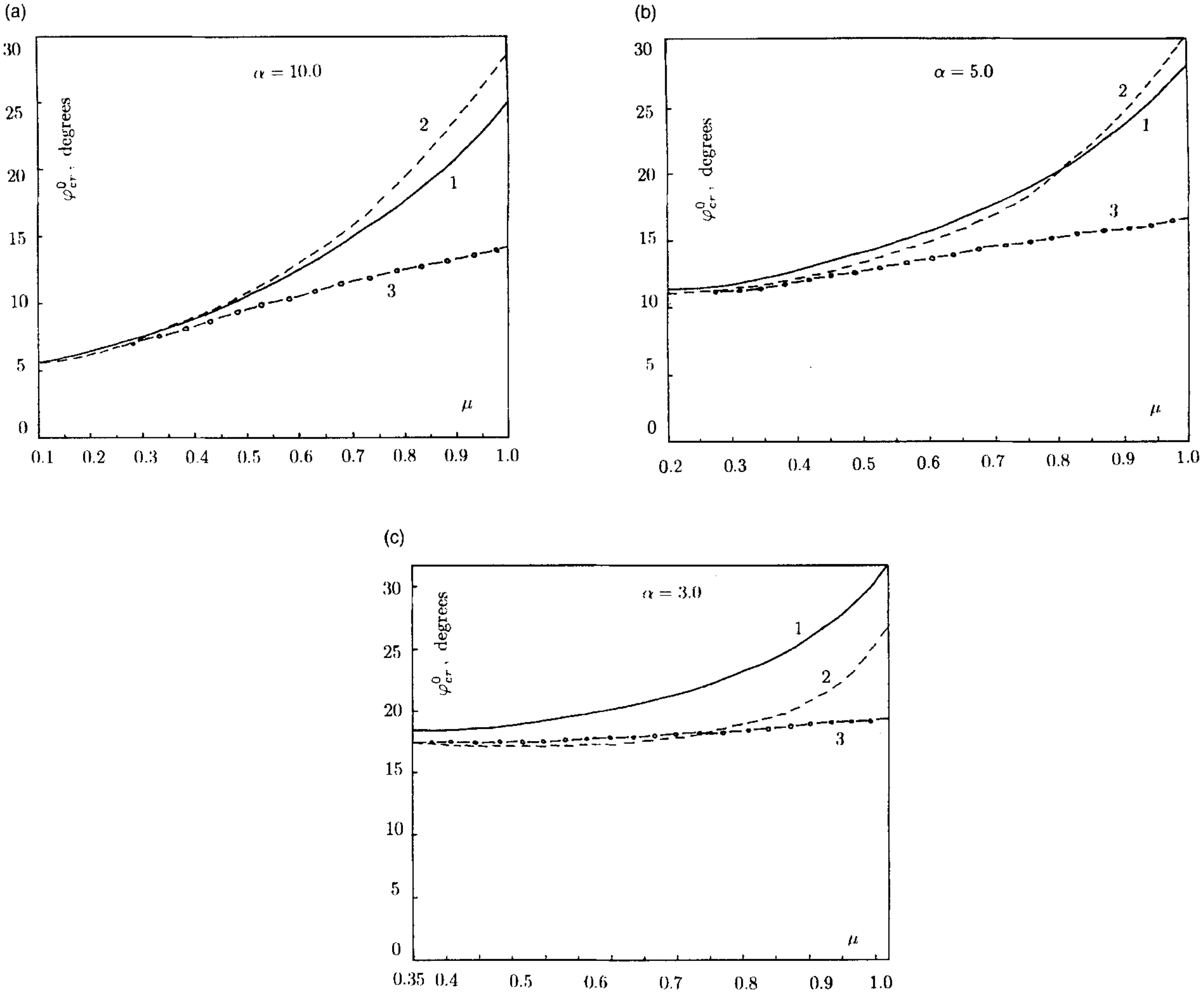

Figure 3. Values of critical angle φ^0_{cr} versus friction coefficient, α: line 1 — exact numerical calculations; line 2 — result predicted by equation (48); line 3 — Formula (49): (a) $\alpha = 10{\cdot}0$; (b) $\alpha = 5{\cdot}0$; (c) $\alpha = 3{\cdot}0$

which follows from equation (19). Since the last integrand has an integrable root-square singularity, the time of overturning

$$t_1(\varphi = \pi/2) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{u_1(\varphi)}}$$

$$= \sqrt{\frac{a}{6g}} \int_0^{\pi/2} \sqrt{\frac{a_1(\varphi)}{(\mu\alpha - 1)\sin\varphi + \alpha(1 - \cos\varphi)}}\, d\varphi \tag{32}$$

exists and takes a finite value.

To verify expression (30), we compare it with a test solution for the thin rod of the length $2h$ posed on the ground without friction

$$u_r(\varphi) = \omega_r^2 = \frac{6g}{h} \frac{1 - \cos\varphi}{1 + 3\sin^2\varphi} \tag{33}$$

Figures 2(a)–(c) show that this is a good approximation for the angular velocity, if $\mu \ll 1$. However approximation (33) has very important defect that does not allow to use it even for extremely small μ, because the time of rotation

$$t_r = \int_0^{\varphi} \frac{d\varphi}{\sqrt{u_r(\varphi)}} \tag{34}$$

becomes indefinite due to a non-integrable singularity of the integrand. Precise numerical calculations were obtained using the Runge–Kutta method applied to equations (8) and (11).

SOME PHYSICAL CONCLUSIONS FOR THE FIRST PHASE OF MOTION

It follows from equation (32) that with fixed parameters α and μ the time of overturning grows with the growth of the width a. Hence, in the class of bodies with the same relative height α the overturning of larger blocks happens more slowly. Thus large constructions are more stable when subjected to an earthquake horizontal ground motion. This fact was known in literature in the case of overturning without sliding.

All the previous considerations imply that the edge of the block, point O, is not separated from the ground, i.e. the body is permanently in a contact with it. This fact should be verified. It follows from equations (7a), (7c), (9c), (14) and (16) that

$$N = \frac{mg(1+\alpha^2)}{c_1(\varphi)} \ddot{\varphi} = \frac{mg(1+\alpha^2)}{2c_1(\varphi)} u_1'(\varphi) \tag{35}$$

so separation is possible if $u_1'(\varphi) < 0$ somewhere on $0 < \varphi < \pi/2$.

Theorem 2. *For the solution given by equation* (33) $u_r'(\varphi) > 0$ $(0 < \varphi < \pi/2)$, *and so jump of the rod in the test problem is impossible.*

Indeed,

$$u_r'(\varphi) = \frac{6g}{h} \frac{1 + 3\sin^2\varphi - 6(1-\cos\varphi)\cos\varphi}{(1+3\sin^2\varphi)^2} \sin\varphi > 0 \tag{36}$$

because numerator of the last expression

$$3\tau^2 - 6\tau + 4, \quad \tau = \cos\varphi \tag{37}$$

is always positive.

Theorem 3. *If the friction coefficient μ is small enough, the block also cannot jump.*

Proof of this theorem is more complicated. For small φ it directly follows from equation (12). If the angle φ is not small it can be shown that for small μ (and so for large $\alpha > 1/\mu$) $u_1(\varphi) \approx u_r(\varphi)$. Hence, statement of the theorem follows from Theorem 2.

Precise numerical investigation has shown that separation in the first phase of the motion indeed does not occur in practice. Only for extremely large $\mu \sim 0{\cdot}99$ we encountered the jump phenomenon.

THE SECOND PHASE OF THE MOTION

Subjected to a horizontal oscillation of the ground surface, direction of the relative sliding of the body can be changed and become, for a period of time, opposite to the initial direction. This phase of the motion is called 'the second phase'.

Let after the block has turned up to angle φ_0 in the first phase the friction force $\bar{F}_f$ change its direction. In this moment the body possesses an angular velocity given by equation (30):

$$\omega_0^2 = u_1^0 = \frac{6g}{a} \frac{(\mu\alpha - 1)\sin\varphi_0 + \alpha(1 - \cos\varphi_0)}{a_1(\varphi_0)} \tag{38}$$

It is clear from the physical point of view that the opposite friction force can in some circumstances return the body back to the initial position. The main attention here will be paid to the question if this returning is possible or the body is overturned in spite of the opposite direction of the friction force.

The second phase of the motion can be described by a first-order linear differential equation involving a function $u_2(\varphi)$ that is a square of the angular velocity:

$$a_2(\varphi)u_2'(\varphi) + 2b_2(\varphi)u_2(\varphi) = 2c_2(\varphi) \tag{39}$$

$$a_2(\varphi) = 1 + \alpha^2 - 3(\cos\varphi - \alpha\sin\varphi)[(\alpha - \mu)\sin\varphi - (\mu\alpha + 1)\cos\varphi] \tag{40a}$$

$$b_2(\varphi) = 3(\sin\varphi + \alpha\cos\varphi)[(\alpha - \mu)\sin\varphi - (\mu\alpha + 1)\cos\varphi] \tag{40b}$$

$$c_2(\varphi) = 3\frac{g}{a}[(\alpha - \mu)\sin\varphi - (\mu\alpha + 1)\cos\varphi] \tag{40c}$$

with the following initial condition

$$\varphi = \varphi_0 \,|\, u_2 = u_1^0 \tag{41}$$

The solution of equation (39) also can be obtained by the Bernoulli method and is given as follows:

$$u_2(\varphi) = u_1^0 \frac{\xi_2(\varphi)}{\xi_2(\varphi_0)} + 2\xi_2(\varphi)\int_{\varphi_0}^{\varphi} \frac{c_2(\varphi)}{\alpha_2(\varphi)\xi_2(\varphi)}\,\mathrm{d}\varphi \tag{42}$$

where the function $\xi_2(\varphi)$ is a solution of homogeneous equation (39) of the following approximate form:

$$\xi_2(\varphi) \approx \frac{1}{a_2(\varphi)} \tag{43}$$

After that integration in equation (42) can be evidently made, and we have, similar to the first phase, an approximate representation

$$u_2(\varphi) = \frac{6g}{a} \frac{W(\varphi, \varphi_0)}{a_2(\varphi)} \tag{44a}$$

$$W(\varphi, \varphi_0) = [(\mu\alpha - 1)\sin\varphi_0 + \alpha(1 - \cos\varphi_0)]\frac{a_2(\varphi_0)}{a_1(\varphi_0)}$$

$$+ \alpha(\cos\varphi_0 - \cos\varphi) - (\mu\alpha + 1)(\sin\varphi - \sin\varphi_0) \tag{44b}$$

valid for small μ. Like in previous consideration, this formula in practice appears to be a good uniform approximation for all values of parameters μ and α except at extremely large value $\mu \sim 1$. Also for these limit values of μ only we could obtain the jump phenomenon in the second phase of motion. Obviously, if $u_2(\varphi) > 0$ for $\varphi_0 < \varphi < \pi/2$ the body will be necessarily overturned, because in this case the time of overturning

$$t_2(\varphi = \pi/2) = \int_0^{\varphi_0} \frac{\mathrm{d}\varphi}{\sqrt{u_1(\varphi)}} + \int_{\varphi_0}^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{u_2(\varphi)}} \tag{45}$$

exists and takes a finite value. Positiveness of $a_2(\varphi)$ can be proved similarly to $a_1(\varphi)$. Therefore, the block returns back when there exists a value of φ such that $W(\varphi, \varphi_0) < 0$, $\varphi_0 < \varphi < \pi/2$.

Let us investigate the possibility for the expression given by equation (44b) to become negative. Minimum value of the function $W(\varphi, \varphi_0)$ respectively φ is reached when $\varphi = \varphi^*$:

$$\tan \varphi^* = \frac{\mu\alpha + 1}{\alpha} \tag{46}$$

independent of φ_0. Substituting equality (46) into equation (44) we have the following:

Theorem 4. *The equation*

$$W(\varphi^*, \varphi_0) = 0 \tag{47}$$

where φ^* is given by (46), separates zones of returning and overturning. If φ_0 is such that $W(\varphi^*, \varphi_0) > 0$, the overturning will certainly occur. If φ_0 is such that $W(\varphi^*, \varphi_0) < 0$, the block returns back to the initial position.

Note that this result does not depend on the size of the body a. For small μ $a_1(\varphi_0) \approx a_2(\varphi_0)$, and the critical value of φ_0, given by equation (47) and separating returning from overturning, can be explicitly derived as follows:

$$\sin \varphi_{cr}^0 = \frac{\mu}{2} \frac{(1 + 1/\mu\alpha)^2}{1 + \sqrt{1 + (\mu + 1/\alpha)^2}} \tag{48}$$

If $0 \leqslant \varphi_0 \leqslant \varphi_{cr}^0$ the body returns back. If $\varphi_{cr}^0 < \varphi_0 < \pi/2$ it overturns. What is the behaviour of the body in the second phase, when in the first phase it has been rotated up to the angle $\varphi_0 = \varphi_{cr}^0$? In general, the following theorem can be proved.

Theorem 5. *If $\varphi_0 = \varphi_{cr}^0$, then for the second phase of the motion the body asymptotically approaches the value $\varphi \to \varphi^*$ given by formula (46).*

Indeed, the time of the motion in the second phase is given by

$$t_2(\varphi) = \int_0^{\varphi_0} \frac{d\psi}{\sqrt{u_1(\psi)}} + \int_{\varphi_0}^{\varphi} \frac{d\psi}{\sqrt{u_1(\psi)}} \tag{49}$$

For $\varphi_0 = \varphi_{cr}^0$ we have $W(\varphi^*, \varphi_0) = 0$, and $\partial W(\varphi, \varphi_0)/\partial\varphi = 0$ if $\varphi = \varphi^*$. Hence it is obvious that in this case

$$u_2(\varphi) \sim (\varphi^* - \varphi)^2, \qquad \varphi \to \varphi^* \tag{50}$$

and the second integral in equation (49) tends to infinity because of a non-integrable singularity of the integrand.

Figures 3(a)–(c) show comparison between the critical angle obtained with precise numerical calculations and the analytical result given by equation (47). Also the values given by simple formula (48), valid for small μ, are reflected on these figures.

An interesting estimation can be made based on the formula (48). Since $\mu\alpha > 1$ due to the natural condition (13), it follows from equation (48) that

$$\sin \varphi_{cr}^0 < \mu \tag{51}$$

A result wonderful of its simplicity. Thus, for small μ we have the following:

Theorem 6. *If in the first phase of the motion the block has reached the angle of rotation φ_0 such that* $\sin \varphi_0 > \mu$, *it cannot come back.*

CONCLUSIONS

(1) As can be seen from formula (48), the critical angle φ_{cr}^{0} is determined principally by the value of the friction coefficient μ and weakly depends upon the relative height of the block α.
(2) Figures 3(a)–(c) show that for small μ estimation (51) for the critical value φ_{cr}^{0} is rather precise, since all the graphics in this region pass near the line $\varphi_{cr}^{0} \sim 180\mu/\pi$ (degrees).
(3) Analytical prediction of the critical angle φ_{cr}^{0} is more precise for large values of the relative height α.
(4) The obtained analytical results possess high accuracy, if μ is out of a vicinity of the extremely possible value $\mu_{max} = 1$. If $\mu \leqslant 0{\cdot}6$ the relative error of analytical approximation is less than 10%.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the financial supports provided under Grant of G.N.F.M. of the Italian Research Council (C.N.R. Italy) and 40% M.U.R.S.T. (Italy).

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